THERMAL STRESSES IN ELASTIC PLATES*

BY

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1. **Introduction.** Thermal changes in an elastic body are accompanied by shifts in the relative positions of the particles composing the body. Such shifts, in general, cannot proceed freely, and thermal stresses are set up in the body. The analytical basis for the determination of such stresses was provided by Duhamel† and Neumann‡ who, starting from certain assumptions, have modified the stress-strain relations of Hooke. The theory based on the law formulated by Duhamel has not been very much developed§ because of the complicated character of the partial differential equations satisfied by the stress components. In particular, there is a notable lack of a careful formulation of the differential equations governing the deflection of elastic plates subjected to nonuniform distribution of temperatures.

Starting with the usual assumptions of the thin plate theory, Nádai|| has developed the differential equation for deflection of a thin elastic plate subjected to a linear distribution of temperature in the direction of the thickness of the plate. In two recent papers Marguerre¶ has considered some related problems, and, on the basis of reasoning essentially similar to that of Nádai, was led to a somewhat more general equation. In the present paper the differential equation governing the deflection of an elastic plate is derived without using the objectionable assumption of the thin plate theory. It will be seen that the differential equations obtained by Nádai and Marguerre are special cases of the more general equation here given. Furthermore, the hypotheses of Nádai and Marguerre that a heated plate will be in a state of plane or generalized plane stress (which form a focal point in their discussion) are not used in this paper. The abrogation of these hypotheses enormously complicates the analysis, but inasmuch as there has been some considerable doubt regarding the validity of the generalized stress assumption, it appears necessary to sacrifice simplicity. The complexity of the situation lies in the nature of the problem itself.

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† J. M. C. Duhamel, *Mémoires . . . par Divers Savants*, vol. 5, 1838, p. 440.

‡ F. E. Neumann, *Vorlesungen über die Theorie der Elasticität der festen Körper*, 1885.

§ Cf. A. E. H. Love, *Mathematical Theory of Elasticity*, 4th edition, 1927, p. 109.

|| A. Nádai, *Elastische Platten*, 1925, p. 268.

¶ K. Marguerre, Zeitschrift für angewandte Mathematik und Mechanik, vol. 15 (1935), pp. 369–372; Ingenieur-Archiv, vol. 8 (1937), pp. 216–228.

2. **General thermo-elastic equations.** According to Duhamel's law the stresses and strains in an elastic body are connected in the following way:

$$(2.1)\qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) = \frac{1+\sigma}{E} X_{ij} - \left(\frac{\sigma}{E}\Theta + \alpha T\right)\delta_{ij}, \quad i, j = 1, 2, 3,$$

where δ_{ij} is the Kronecker delta, $\Theta = \sum_{i=1}^{3} X_{ii}$, T is the prescribed temperature at any point of the body, and α is a constant depending on the physical properties of the material. Young's modulus E, and Poisson's ratio σ are regarded as independent of the temperature.*

Since the equations of equilibrium are deduced with no reference to the law connecting stresses and strains, they remain valid in this case.† They are

$$(2.2)\qquad \sum_{j=1}^{3} \frac{\partial X_{ij}}{\partial x_j} = 0, \qquad i = 1, 2, 3,$$

where the stress components X_{ij} must satisfy, on the boundary of the solid, the following conditions:

$$(2.3)\qquad \sum_{j=1}^{3} X_{ij} \cos(x_j, n) = X_{ni}, \qquad i = 1, 2, 3.$$

It is well known that the satisfaction of (2.2) and (2.3) does not guarantee a physically realizable system of stresses. The additional conditions which form a connecting link between (2.1) and (2.2) are the compatibility equations of St. Venant‡ which demand, in effect, that the displacements u_i in a simply connected region be single-valued functions. Substitution of (2.1) in St. Venant's compatibility equations gives, after some reduction, the desired equations of connection:

$$(2.4)\qquad \nabla^2 X_{ij} + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x_i \partial x_j} = -\frac{\alpha E}{1-\sigma}\nabla^2 T \delta_{ij} - \frac{\alpha E}{1+\sigma}\frac{\partial^2 T}{\partial x_i \partial x_j}, \qquad i, j = 1, 2, 3,$$

where Θ satisfies the equation

$$(2.5)\qquad \nabla^2 \Theta = -\frac{2\alpha E}{1-\sigma}\nabla^2 T$$

and

* For some critical remarks regarding the applicability of these equations in practice, see a paper by J. N. Goodier, Philosophical Magazine, vol. 23 (1937), pp. 1017–1032.

† A. E. H. Love, loc. cit., p. 100.

‡ A. E. H. Love, loc. cit., p. 49, equations (25).

$$\nabla^2 \equiv \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i \partial x_i} \cdot$$

The equations (2.4) reduce, as they should, to the compatibility equations of Beltrami* when T is set equal to a constant. The sets of equations (2.2), (2.3), and (2.4) give a unique determination of stresses.

In order to avoid the use of the subscripts on the variables u_3 and x_3, which figure prominently in the remainder of this paper, the letters w and z will be used in their stead. The notation adopted at this point agrees with that of Love's treatise.

One can readily obtain a set of equations in the displacements u, v, and w by substituting the expressions for the stress components from (2.1) in (2.2). The resulting equations† are

$$(2.6) \qquad \nabla^2 w = \frac{2\alpha(1+\sigma)}{1-2\sigma}\frac{\partial T}{\partial z} - \frac{1}{1-2\sigma}\frac{\partial \Delta}{\partial z},$$

and two similar equations for u and v, where

$$\Delta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \cdot$$

It will be observed, with reference to (2.1), that

$$(2.7) \qquad \Delta = \frac{1-2\sigma}{E}\Theta + 3\alpha T.$$

The foregoing equations give a unique characterization of the behavior of a simply connected elastic body subjected to heat, and involve no simplifying assumptions in regard to the geometrical properties of the body. In deriving the thermo-elastic plate equation, both Nádai and Marguerre assume that the plate is so thin that one is permitted to write

$$(2.8) \qquad u = -z\frac{\partial w_0}{\partial x}, \qquad v = -z\frac{\partial w_0}{\partial y},$$

where w_0 denotes the deflection of the "middle surface" of the plate, and is a function of x and y only. They further assume that the plate is in a state of plane stress, so that one is permitted to set

$$(2.9) \qquad Z_z = 0$$

throughout the thickness of the plate.

* A. E. H. Love, loc. cit., p. 135.

† Cf. S. Timoshenko, *Theory of Elasticity*, 1934, p. 205.

A number of investigators in the theory of elasticity have objected to the simplifying assumption (2.9), and have attempted to justify it on the basis of something more satisfactory than intuitive feeling or experimental grounds. The latest of such attempts is that of Southwell* who investigates the distribution of stress along the edges of the plate where it is assumed that the plate is in a state of generalized plane stress. His results, although not quite conclusive, point to the fact that in order to maintain a state of generalized plane stress one is obliged to apply a complicated distribution of stresses on the edges of the plate of a type which is not likely to be realized in practice. The derivation of the differential equation for the deflection of the middle surface of an elastic plate given in the next section makes no use of the simplifying assumptions (2.8) and (2.9), and thus appears to be applicable to thick as well as thin plates.

3. **Thermo-elastic plate equation.** The two-dimensional Laplacian operator $\partial^2/\partial x^2+\partial^2/\partial y^2$ will be denoted here by the symbol ∇_1^2, so that

$$\nabla_1^2 w = \nabla^2 w - \frac{\partial^2 w}{\partial z^2} \cdot$$

Substituting (2.7) in (2.6) gives

$$\nabla_1^2 w = -\alpha \frac{\partial T}{\partial z} - \frac{1}{E}\frac{\partial \Theta}{\partial z} - \frac{\partial^2 w}{\partial z^2} \cdot \tag{3.1}$$

Differentiating the stress-strain relations (2.1) for e_{33} with respect to z and substituting the resulting value of $\partial^2 w/\partial z^2$ in (3.1) gives

$$\nabla_1^2 w = -2\alpha \frac{\partial T}{\partial z} + \frac{\sigma - 1}{E}\frac{\partial \Theta}{\partial z} - \frac{\sigma + 1}{E}\frac{\partial Z_z}{\partial z} \cdot \tag{3.2}$$

But the equations (2.4) with $i=j=3$ give

$$\frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial z^2} = -\frac{\alpha E}{1-\sigma}\nabla^2 T - \frac{\alpha E}{1+\sigma}\frac{\partial^2 T}{\partial z^2} - \nabla^2 Z_z, \tag{3.3}$$

and from (2.5),

$$\begin{aligned} \nabla_1^2 \Theta &= -\frac{2\alpha E}{1-\sigma}\nabla^2 T - \frac{\partial^2 \Theta}{\partial z^2} \\ &= -\alpha E \nabla_1^2 T + (1+\sigma)\nabla^2 Z_z, \end{aligned} \tag{3.4}$$

where the last step results from the substitution of the value of $\partial^2\Theta/\partial z^2$ from (3.3).

* R. V. Southwell, Philosophical Magazine, vol. 21 (1936), pp. 201–215.

Operating on (3.2) with ∇_1^2 gives

$$\nabla_1^4 w = -2\alpha \frac{\partial}{\partial z}\nabla_1^2 T + \frac{\sigma - 1}{E}\frac{\partial}{\partial z}\nabla_1^2\Theta - \frac{\sigma+1}{E}\frac{\partial}{\partial z}\nabla_1^2 Z_z, \tag{3.5}$$

and substituting in (3.5) and (3.4) and simplifying, we obtain

$$\nabla_1^4 w = -\alpha(1+\sigma)\nabla_1^2\frac{\partial T}{\partial z} + \frac{(1+\sigma)(\sigma-2)}{E}\frac{\partial}{\partial z}\nabla_1^2 Z_z + \frac{\sigma^2-1}{E}\frac{\partial^3 Z_z}{\partial z^3}. \tag{3.6}$$

The equation of the middle plane is obtained from (3.6) by setting $z=0$. Thus one can write

$$\begin{aligned}\nabla_1^4 w_0 = &-\alpha(1+\sigma)\nabla_1^2\left(\frac{\partial T}{\partial z}\right)_0 + \frac{(1+\sigma)(\sigma-2)}{E}\nabla_1^2\left(\frac{\partial Z_z}{\partial z}\right)_0 \\ &+ \frac{\sigma^2-1}{E}\left(\frac{\partial^3 Z_z}{\partial z^3}\right)_0,\end{aligned} \tag{3.7}$$

where the zero subscripts denote that the values of the expressions affected are calculated by setting $z=0$.

It is interesting to note that if one assumes with Nádai*

$$T = T_0(x, y) + zT_1(x, y)$$

and $Z_z=0$, the equation (3.7) reduces directly to that obtained by Nádai without invoking his additional assumption (2.8).

Equation (3.7) contains in the right-hand member the unknown function Z_z, the determination of which is given in the following section. It will be seen that in the case of thin plates, for a suitably restricted T, Z_z is quite small. This affords some justification for the assumption (2.9) of earlier investigators.

4. **The determination of Z_z.** The differential equation satisfied by Z_z is obtained by setting $i=j=3$ in (2.4), operating with the Laplacian operator and noting from (2.5) that

$$\nabla^2\frac{\partial^2\Theta}{\partial z^2} = -\frac{2\alpha E}{1-\sigma}\nabla^2\frac{\partial^2 T}{\partial z^2}.$$

The result is

$$\nabla^4 Z_z = \frac{\alpha E}{1-\sigma}\left(\frac{\partial^2}{\partial z^2}\nabla^2 T - \nabla^4 T\right). \tag{4.1}$$

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* Nádai, loc. cit., p. 265.

Denote the thickness of the plate by $2h$, and let its faces be given by $z=\pm h$; then it follows from (2.3) that

$$Z_z = Z_n, \quad X_z = X_n, \quad Y_z = Y_n, \quad \text{on} \quad z = \pm h.$$

If the faces of the plate are free from external loads,* so that the thermal stresses are the only ones under consideration, it is clear that

$$\begin{aligned} Z_z(x, y, \pm h) &= 0, \\ Z_x(x, y, \pm h) &= 0, \\ Z_y(x, y, \pm h) &= 0. \end{aligned} \tag{4.2}$$

From (4.2) and the equations (2.2) it follows that

$$\frac{\partial Z_z}{\partial z} = 0, \quad \text{on} \quad z = \pm h. \tag{4.3}$$

Thus in addition to satisfying (4.1), Z_z together with $\partial Z_z/\partial z$ must vanish on the faces of the plate.

If the plate is so large that one is justified in regarding the problem as two-dimensional, the equation (4.1) together with the boundary conditions (4.2) and (4.3) determine Z_z uniquely. In particular, if the right-hand member of (4.1) is zero (which is certainly the case when T is steady) and if the plate is infinite, the only solution of the problem is $Z_z=0$. This suggests that in the case of finite plates of thickness small compared with the linear dimensions in the x and y directions, Z_z cannot be very great. In fact, let Z_z be expanded in an infinite series in powers of z so that

$$Z_z = \sum_{j=0}^{\infty} a_j(x, y)z^j,$$

where the coefficients a_j are unknown functions of x and y. It will be seen presently that the determination of these unknown functions can be made to depend on the solution of an infinite system of partial differential equations of a complicated sort. In order to make the problem tractable, it is desirable to introduce at this point a simplifying hypothesis† of quite mild nature. It will be assumed that Z_z is adequately represented by a polynomial in z of the form

$$Z_z = \sum_{j=0}^{n} a_j(x, y)z^j, \tag{4.4}$$

* If the plate is subjected to a normal pressure p, the first of the boundary conditions reads $Z_z(x, y, h) = -p$, $Z_z(x, y, -h)=0$. The introduction of p produces no essential complications in the argument that is to follow.

† See in this connection §8 below.

where n is finite, but can be chosen arbitrarily large. This amounts to assuming that, for sufficiently large n, the remainder in the Taylor expansion for Z_z is negligibly small. In order to make the argument clear and not to complicate unduly the resulting equations, the discussion will be confined to a distribution of the temperature of the form*

$$T = \sum_{k=0}^{3} T_k(x, y)z^k, \tag{4.5}$$

which includes the important linear case of Nádai,† as well as the cases considered by Timoshenko.‡ The extension to a more complicated distribution of temperature is obvious and involves little more than the change of the indices of summation.

Noting the operator identity

$$\nabla^4 \equiv \nabla_1^4 + 2\frac{\partial^2}{\partial z^2}\nabla_1^2 + \frac{\partial^4}{\partial z^4},$$

where

$$\nabla_1^4 \equiv \nabla_1^2\nabla_1^2,$$

and calculating with its aid $\nabla^4 Z_z$ defined by (4.4) gives

$$\nabla^4 Z_z = \sum_{j=0}^{n} \left[z^j\nabla_1^4 a_j + 2j(j-1)z^{j-2}\nabla_1^2 a_j + j(j-1)(j-2)(j-3)a_j z^{j-4}\right].$$

The right-hand member of (4.1) upon substituting for T from (4.5) becomes

$$\frac{\alpha E}{1-\sigma}\left(\frac{\partial^2}{\partial z^2}\nabla^2 T - \nabla^4 T\right) = -\frac{\alpha E}{1-\sigma}\sum_{k=0}^{3}\left[z^k\nabla_1^4 T_k + k(k-1)z^{k-2}\nabla_1^2 T_k\right],$$

so that (4.1) gives the identity in z,

$$\sum_{j=0}^{n} \left[z^j\nabla_1^4 a_j + 2j(j-1)z^{j-2}\nabla_1^2 a_j + j(j-1)(j-2)(j-3)a_j z^{j-4}\right]$$
$$= -\frac{\alpha E}{1-\sigma}\sum_{k=0}^{3}\left[z^k\nabla_1^4 T_k + k(k-1)z^{k-2}\nabla_1^2 T_k\right].$$

Equating the coefficients of like powers of z on both sides of this equation gives a system of partial differential equations to be satisfied by the a_j, namely,

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* This is not an assumption since we prescribe T. Besides the space variables x and y, T_k may contain the time variable as a parameter.

† Nádai, loc. cit., p. 265.

‡ S. Timoshenko, loc. cit., p. 207.

$$(4.6) \qquad \nabla_1^4 \alpha_k + 2\nabla_1^2 \alpha_{k+2} + \alpha_{k+4} = \nabla_1^2 (\nabla_1^2 \tau_k + \tau_{k+2}), \qquad k = 0, 1, 2, \cdots, n,$$

where

$$\alpha_k \equiv k! a_k, \quad \tau_k \equiv k! c T_k, \quad \alpha_i \equiv 0, \text{ if } i > n, \quad \tau_j \equiv 0, \text{ if } j > 3,$$

and

$$c \equiv -\frac{\alpha E}{1-\sigma}.$$

Two important observations regarding the structure of the system (4.6) are in order:

(i) The system can be broken up into two independent systems of identical forms, one of which depends on the α_k with even subscripts, while the other depends on the α_k with odd subscripts.

(ii) Every α_k in the first system can be expressed explicitly in terms of α_0 and α_2, whereas α_1 and α_3 determine every α_k of the second system. The last assertion becomes clear when the first $n-3$ of equations (4.6) are solved for α_{k+4} to give

$$(4.7) \qquad \alpha_{k+4} = \nabla_1^4 (\tau_k - \alpha_k) + \nabla_1^2 (\tau_{k+2} - 2\alpha_{k+2}), \qquad k = 0, 1, 2, \cdots, n-4.$$

Inasmuch as the system of equations associated with the even subscripts on the α_k's is of the same form as that for the odd ones, it will suffice to consider only one of them.

Thus, setting $k=0$ in (4.7) gives

$$\alpha_4 = \nabla_1^4 (\tau_0 - \alpha_0) + \nabla_1^2 (\tau_2 - 2\alpha_2).$$

Substituting this value of α_4 in α_6 obtained from (4.7) by taking $k=2$ gives

$$\alpha_6 = -2\nabla_1^6 (\tau_0 - \alpha_0) - \nabla_1^4 (\tau_2 - 3\alpha_2).$$

Introducing these expressions for α_6 and α_4 in the right-hand member of (4.7) with $k=4$ gives α_8 entirely in terms of α_0 and α_2. The continuation of this process of successive substitutions gives the formula

$$(4.8) \qquad \alpha_{2k} = (-1)^k [(k-1)\nabla_1^{2k} (\tau_0 - \alpha_0) + \nabla_1^{2k-2} (\tau_2 - k\alpha_2)], \qquad k = 2, 3, \cdots, N,$$

where $N \equiv (n/2)$, the greatest integer contained in $n/2$. Thus, it is merely necessary to obtain the solutions for α_0 and α_2 in order to determine completely the remaining α_{2k}. The corresponding result for the odd α_k is

$$(4.9) \qquad \alpha_{2k+1} = (-1)^k [(k-1)\nabla_1^{2k} (\tau_1 - \alpha_1) + \nabla_1^{2k-2} (\tau_3 - k\alpha_3)], \qquad k = 2, 3, \cdots, M,$$

where $M \equiv ((n-1)/2)$, the greatest integer contained in $(n-1)/2$.

5. **The determination of a_0, a_1, a_2, a_3.** The boundary conditions (4.2) and (4.3) impose some restrictions on the functions α_k, and the nature of these restrictions will be investigated next. Substituting (4.4) in (4.2) and (4.3) gives

$$\begin{aligned} a_0 + a_1 h + a_2 h^2 + a_3 h^3 &= -\left[a_4 h^4 + \cdots + a_n h^n\right], \\ a_0 - a_1 h + a_2 h^2 - a_3 h^3 &= -\left[a_4 h^4 - \cdots + (-1)^n a_n h^n\right], \\ a_1 + 2a_2 h + 3a_3 h^2 &= -\left[4a_4 h^3 + \cdots + n a_n h^{n-1}\right], \\ a_1 - 2a_2 h + 3a_3 h^2 &= -\left[-4a_4 h^3 + \cdots + (-1)^{n-1} n a_n h^{n-1}\right]. \end{aligned}$$

The determinant of the coefficients of the a_j in the left-hand members of this system of equations is equal to $-16h^4$; so one can solve for a_j, $(j=0, 1, 2, 3)$, in terms of the remaining ones. If $n \geqq 4$, the solution for a_0 is*

$$a_0 = \frac{1}{16h^4} \sum_{j=4}^{n} a_j h^{j-1} \begin{vmatrix} h & h & h^2 & h^3 \\ (-1)^j h & -h & h^2 & -h^3 \\ j & 1 & 2h & 3h^2 \\ (-1)^{j-1} j & 1 & -2h & 3h^2 \end{vmatrix}.$$

It is clear that the value of the coefficient of h^{j-1} corresponding to odd values of j is zero. Setting $j=2k$ gives, after some elementary reductions,

$$\alpha_0 = a_0 = \sum_{k=2}^{N} (k-1) h^{2k} a_{2k} = \sum_{k=2}^{N} \frac{(k-1)}{(2k)!} h^{2k} \alpha_{2k}. \tag{5.1}$$

Similarly solving for a_2, a_1, and a_3 one obtains

$$\alpha_2 = 2a_2 = -2\sum_{k=2}^{N} k h^{2k-2} a_{2k} = -2\sum_{k=2}^{N} \frac{k}{(2k)!} h^{2k-2} \alpha_{2k}, \tag{5.2}$$

$$\alpha_1 = a_1 = \sum_{k=2}^{M} (k-1) h^{2k} a_{2k+1} = \sum_{k=2}^{M} \frac{k-1}{(2k+1)!} h^{2k} \alpha_{2k+1}, \tag{5.3}$$

$$\alpha_3 = 6a_3 = -6\sum_{k=2}^{M} k h^{2k-2} a_{2k+1} = -6\sum_{k=2}^{M} \frac{k}{(2k+1)!} h^{2k-2} \alpha_{2k+1}. \tag{5.4}$$

It is thus seen that the equations arising from the boundary conditions also fall into two independent groups associated with even and odd subscripts on a_j, respectively. The important consequence of this observation lies in the fact that one is led to two independent systems of differential equations and boundary conditions, where the functions a_{2k} are determined by T_0 and T_2, and the a_{2k+1} by T_1 and T_3.

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* If $n<4$, $a_0=a_1=a_2=a_3=0$. See in this connection §7 and §8.

Inasmuch as the two systems of equations are of the same form, the discussion to follow is confined to the system depending on the even subscripts.

Referring to (4.6) and (4.8), and remembering that $n \geqq 4$, one sees that this latter system is

$$\alpha_{2k} = (-1)^k[(k-1)\nabla_1^{2k}(\tau_0 - \alpha_0) + \nabla_1^{2k-2}(\tau_2 - k\alpha_2)], \quad k = 2, 3, \cdots, N, \tag{5.5}$$

$$\nabla_1^2(\nabla_1^2\alpha_{2N-2} + 2\alpha_{2N}) = \nabla_1^4\tau_{2N-2}, \tag{5.6}$$

$$\nabla_1^4\alpha_{2N} = 0. \tag{5.7}$$

The right-hand member in (5.6) vanishes if $N > 2$. Consider first the system (5.5), (5.6), (5.7) when $N \geqq 3$. Using (5.5) with $k = N$ and $k = N-1$, and substituting the resulting values for α_{2k} in (5.6) and (5.7) give, after some algebraic reductions,

$$\nabla_1^{2N}[N\nabla_1^2\alpha_0 + (N+1)\alpha_2] = \nabla_1^{2N}(N\nabla_1^2\tau_0 + \tau_2), \tag{5.8}$$

$$\nabla_1^{2N+2}[(N-1)\nabla_1^2\alpha_0 + N\alpha_2] = \nabla_1^{2N+2}[(N-1)\nabla_1^2\tau_0 + \tau_2]. \tag{5.9}$$

The differential equations for α_0 and α_2 are obtained by operating on (5.8) with ∇_1^2 and solving for $\nabla_1^{2N+4}\alpha_0$ and $\nabla_1^{2N+2}\alpha_2$. The result is

$$\nabla_1^{2N+2}\alpha_2 = \nabla_1^{2N+2}\tau_2, \tag{5.10}$$

$$\nabla_1^{2N+4}\alpha_0 = \nabla_1^{2N+2}(\nabla_1^2\tau_0 - \tau_2). \tag{5.11}$$

These equations will be solved by a device of constructing a sequence of differential equations of lower orders which α_0 and α_2 must satisfy.

Substitution of (5.5) in the boundary conditions (5.1) and (5.2) gives

$$\alpha_0 = \sum_{k=2}^{N} \frac{(-1)^k(k-1)}{(2k)!} h^{2k}[(k-1)\nabla_1^2\ (\tau_0 - \alpha_0) + \nabla_1^{2k-2}(\tau_2 - k\alpha_2)], \tag{5.12}$$

$$\alpha_2 = -\sum_{k=2}^{N} \frac{(-1)^k 2k}{(2k)!} h^{2k-2}[(k-1)\nabla_1^{2k}(\tau_0 - \alpha_0) + \nabla_1^{2k-2}(\tau_2 - k\alpha_2)]. \tag{5.13}$$

Operating on (5.12) with ∇_1^{2N} produces

$$\begin{aligned}
\nabla_1^{2N}\alpha_0 &= \sum_{k=2}^{N} \frac{(-1)^k(k-1)}{(2k)!} h^{2k}[(k-1)\nabla_1^{2k+2N}(\tau_0 - \alpha_0) + \nabla_1^{2k+2N-2}(\tau_2 - k\alpha_2)] \\
&= \sum_{k=2}^{N} \frac{(-1)^k(k-1)}{(2k)!} h^{2k}[(k-1)\nabla_1^{2k+2N}(\tau_0 - \alpha_0) + k\nabla_1^{2k+2N-2}(\tau_2 - \alpha_2) \\
&\qquad - (k-1)\nabla_1^{2k+2N-2}\tau_2] \\
&= \sum_{k=2}^{N} \frac{(-1)^k(k-1)}{(2k)!} h^{2k}\nabla_1^{2k-4}\{(k-1)[\nabla_1^{2N+2}(\nabla_1^2\tau_0 - \tau_2) - \nabla_1^{2N+4}\alpha_0] \\
&\qquad + k[\nabla_1^{2N+2}\tau_2 - \nabla_1^{2N+2}\alpha_2]\},
\end{aligned}$$

which vanishes since the terms in the brackets vanish by (5.10) and (5.11). Thus

$$\nabla_1^{2N}\alpha_0 = 0.$$

In precisely the same way it is shown that

$$\nabla_1^{2N}\alpha_2 = 0.$$

Referring to (5.8) and (5.9) one sees that these last two equations demand that

$$(5.14)\qquad N\nabla_1^{2N+2}\tau_0 + \nabla_1^{2N}\tau_2 = 0, \qquad (N-1)\nabla_1^{2N+4}\tau_0 + \nabla_1^{2N+2}\tau_2 = 0.$$

Therefore*

$$(5.15)\qquad \nabla_1^{2N+4}\tau_0 = 0, \qquad \nabla_1^{2N+2}\tau_2 = 0.$$

Again calculating $\nabla_1^{2N-2}\alpha_0$ and $\nabla_1^{2N-2}\alpha_2$ from (5.12) and (5.13), and taking account of the relations just found furnishes two equations for α_0 and α_2 of lower order than the preceding ones, namely,

$$\nabla_1^{2N-2}\alpha_0 = \frac{h^4}{4!}\nabla_1^{2N}(\nabla_1^2\tau_0 + \tau_2), \qquad \nabla_1^{2N-2}\alpha_2 = -\frac{h^3}{3!}\nabla_1^{2N}(\nabla_1^2\tau_0 + \tau_2).$$

The result of operating on (5.12) and (5.13) with ∇_1^{2N-4} and noting the two equations just found is

$$\nabla_1^{2N-4}\alpha_0 = \frac{h^4}{4!}\nabla_1^{2N-2}(\nabla_1^2\tau_0 + \tau_2) + \frac{h^6}{6!}\nabla_1^{2N}(6\nabla_1^2\tau_0 + 8\tau_2),$$

$$\nabla_1^{2N-4}\alpha_2 = -\frac{h^2}{3!}\nabla_1^{2N-2}(\nabla_1^2\tau_0 + \tau_2) - \frac{h^4}{6!}\nabla_1^{2N}(28\nabla_1^2\tau_0 + 34\tau_2).$$

A little reflection will show that a continuation of this process of operating successively on α_0 and α_2 with ∇_1^{2N-2p}, $(p=0, 1, 2, \cdots, N)$, will yield, after N operations, expressions for α_0 and α_2 of the type

$$(5.16)\qquad \alpha_0 = \sum_{n=2}^{N+1} A_{2n}h^{2n}, \qquad \alpha_2 = \sum_{n=2}^{N+1} B_{2n-2}h^{2n-2},$$

where

$$(5.17)\qquad \begin{aligned} A_{2n} &= \lambda_{2n}\nabla_1^{2n}\tau_0 + \mu_{2n}\nabla_1^{2n-2}\tau_2, \\ B_{2n} &= \rho_{2n}\nabla_1^{2n+2}\tau_0 + \sigma_{2n}\nabla_1^{2n}\tau_2, \end{aligned}$$

in which λ, μ, ρ, and σ are constants.

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* For significance of this see §8 below.

The device just outlined can be used successfully to determine the α_0 and α_2 for any $N \geqq 3$, but it is simpler to calculate the functions A_{2n} and B_{2n} by the method of undetermined coefficients since the forms of α_0 and α_2 are now known. This method is followed in the next section where a set of compact formulas is developed for the determination of these functions.

An argument in every respect similar to that just used shows that if $M \geqq 3$,

$$\alpha_1 = \sum_{n=2}^{M+1} \overline{A}_{2n} h^{2n}, \qquad \alpha_3 = \sum_{n=2}^{M+1} \overline{B}_{2n-2} h^{2n-2}, \tag{5.18}$$

where $\overline{A}_{2n}$ and $\overline{B}_{2n}$ are of the form (5.17) with τ_0 and τ_2 replaced by τ_1 and τ_3, respectively. The functions τ_1 and τ_3 satisfy the equations

$$\nabla_1^{2M}(M\nabla_1^2\tau_1 + \tau_3) = 0, \qquad \nabla_1^{2M+4}\tau_1 = 0, \qquad \nabla_1^{2M+2}\tau_3 = 0, \tag{5.19}$$

which correspond to (5.14) and (5.15).

6. **Recursion formulas for A_k and B_k.** Substituting the expressions for α_0 and α_2 from (5.16) in the right-hand member of (5.12) and rearranging gives

$$\begin{aligned}\alpha_0 = \sum_{k=2}^{N} \frac{(-1)^k(k-1)}{(2k)!} h^{2k}\Bigg[(k-1)\nabla_1^{2k}\left(\tau_0 - \sum_{j=2}^{N+1} A_{2j}h^{2j}\right) \\ + \nabla_1^{2k-2}\left(\tau_2 - k\sum_{j=1}^{N} B_{2j}h^{2j}\right)\Bigg] \\ = \sum_{k=2}^{N} \frac{(-1)^k(k-1)}{(2k)!}\Bigg\{\left[(k-1)\nabla_1^{2k-2}(\nabla_1^2\tau_0 - \tau_2) + k\nabla_1^{2k-2}\tau_2\right]h^{2k} \\ - k\nabla_1^{2k-2}B_2h^{2k+2} - \sum_{j=2}^{N} h^{2k+2j}\nabla_1^{2k-2}\left[(k-1)\nabla_1^2A_{2j} + kB_{2j}\right] \\ - (k-1)\nabla_1^{2k}A_{2N+2}h^{2N+2k+2}\Bigg\}.\end{aligned}$$

This expression can be written as

$$\alpha_0 = \sum_{k=2}^{N} \frac{(-1)^{k-1}(k-1)}{(2k)!} \sum_{j=0}^{N+1} h^{2k+2j}\nabla_1^{2k-2}\left[(k-1)\nabla_1^2A_{2j} + kB_{2j}\right], \tag{6.1}$$

where the following definitions are introduced:

$$\nabla_1^{2k} A_0 = \nabla_1^{2k-2}(-\nabla_1^2 \tau_0 + \tau_2), \quad \nabla_1^{2k-2} B_0 = -\nabla_1^{2k-2}\tau_2, \tag{6.2}$$
$$\nabla_1^{2k} A_2 = 0, \qquad \nabla_1^{2k-2} B_{2N+2} = 0.$$

A similar substitution in the right-hand member of (5.13) gives

$$\alpha_2 = \sum_{k=2}^{N} \frac{(-1)^k 2k}{(2k)!} \sum_{j=0}^{N+1} h^{2k+2j-2} \nabla_1^{2k-2}[(k-1)\nabla_1^2 A_{2j} + kB_{2j}]. \tag{6.3}$$

In order to collect the coefficients of powers of h, set $k+j=r$; then (6.1) and (6.3) become*

$$(6.4)\quad \begin{aligned} \alpha_0 &= \sum_{r=2}^{2N+1} h^{2r} \sum_{j=0}^{r-2} (-1)^{r-j-1} \frac{r-j-1}{(2r-2j)!} \nabla_1^{2r-2j-2}[(r-j-1)\nabla_1^2 A_{2j} + (r-j)B_{2j}], \\ \alpha_2 &= \sum_{r=2}^{2N+1} h^{2r-2} \sum_{j=0}^{r-2} (-1)^{r-j} \frac{2(r-j)}{(2r-2j)!} \nabla_1^{2r-2j-2}[(r-j-1)\nabla_1^2 A_{2j} + (r-j)B_{2j}]. \end{aligned}$$

Equating the coefficients of like powers of h in (5.16) and (6.4) gives the desired recursion formulas

$$(6.5)\quad \begin{aligned} A_{2r} &= \sum_{j=0}^{r-2} (-1)^{r-j-1} \frac{r-j-1}{(2r-2j)!} \nabla_1^{2r-2j-2}[(r-j-1)\nabla_1^2 A_{2j} + (r-j)B_{2j}], \\ B_{2r-2} &= \sum_{j=0}^{r-2} (-1)^{r-j} \frac{2(r-j)}{(2r-2j)!} \nabla_1^{2r-2j-2}[(r-j-1)\nabla_1^2 A_{2j} + (r-j)B_{2j}], \\ & \qquad r = 2, 3, \cdots, N+1. \end{aligned}$$

Referring to (6.2) it is clear that the functions A_{2k} and B_{2k} can be expressed for any $N \geqq 3$ in terms of the prescribed functions τ_0 and τ_2, so that α_0 and α_2, and hence α_{2k}, are completely determined.

The recursion formulas for $\overline{A}_{2n}$ and $\overline{B}_{2n}$ in (5.18), deduced by precisely the same method, are

$$(6.6)\quad \begin{aligned} \overline{A}_{2r} &= \sum_{j=0}^{r-2} (-1)^{r-j-1} \frac{r-j-1}{(2r-2j+1)!} \nabla_1^{2r-2j-2}[(r-j-1)\nabla_1^2 \overline{A}_{2j} + (r-j)\overline{B}_{2j}], \\ \overline{B}_{2r-2} &= \sum_{j=0}^{r-2} (-1)^{r-j} \frac{6(r-j)}{(2r-2j+1)!} \nabla_1^{2r-2j-2}[(r-j-1)\nabla_1^2 \overline{A}_{2j} + (r-j)\overline{B}_{2j}], \end{aligned}$$

where

$$\nabla_1^{2k} \overline{A}_0 = \nabla_1^{2k-2}(-\nabla_1^2 \tau_1 + \tau_3), \quad \nabla_1^{2k-2} \overline{B}_0 = -\nabla_1^{2k-2}\tau_3,$$
$$\nabla_1^{2k} \overline{A}_2 = 0, \qquad \nabla_1^{2k-2} \overline{B}_{2M+2} = 0.$$

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* This amounts to rearranging the finite double sums in (6.1) and (6.2) so that summations proceed first along the diagonals.

7. **Special case** $N=2$, $M=2$. The determination of Z_z was carried out under the assumption that $N\geqq 3$ and $M\geqq 3$, so that the degree of the polynomial (4.4) was assumed to be greater than 5. It follows from the boundary conditions* that unless $n>3$ the only possible solution for Z_z, of the form (4.4), is $Z_z=0$. It remains to consider the case when $n=4$ or $n=5$, that is, $N=M=2$.

Referring to (5.5), (5.6), and (5.7), one sees that the system of equations associated with even subscripts in this case is

$$\alpha_4 = \nabla_1^4\tau_0 + \nabla_1^2\tau_2 - \nabla_1^4\alpha_0 - 2\nabla_1^2\alpha_2, \tag{7.1}$$

$$\nabla_1^4\alpha_2 + 2\nabla_1^2\alpha_4 = \nabla_1^4\tau_2, \tag{7.2}$$

$$\nabla_1^4\alpha_4 = 0, \tag{7.3}$$

and the boundary conditions (5.1) and (5.2) become

$$\alpha_0 = \frac{h^4}{4!}\alpha_4, \qquad \alpha_2 = -\frac{h^2}{3!}\alpha_4. \tag{7.4}$$

From (7.4) and (7.3) it follows that

$$\nabla_1^4\alpha_0 = 0, \qquad \nabla_1^4\alpha_2 = 0. \tag{7.5}$$

Hence (7.2) gives

$$\nabla_1^2\alpha_4 = \frac{1}{2!}\nabla_1^4\tau_2. \tag{7.6}$$

Since $\nabla_1^4\alpha_2=0$, (7.6) requires that

$$\nabla_1^6\tau_2 = 0. \tag{7.7}$$

Calculating $\nabla_1^4\alpha_4$ from (7.1) and making use of (7.7) gives

$$\nabla_1^8\tau_0 = 0. \tag{7.8}$$

But from (5.12) and (5.13),

$$\alpha_0 = \frac{h^4}{4!}(\nabla_1^4\tau_0 + \nabla_1^2\tau_2 - \nabla_1^4\alpha_0 - 2\nabla_1^2\alpha_2), \tag{7.9}$$

and

$$\alpha_2 = -\frac{h^2}{3!}(\nabla_1^4\tau_0 + \nabla_1^2\tau_2 - \nabla_1^4\alpha_0 - 2\nabla_1^2\alpha_2). \tag{7.10}$$

Calculating $\nabla_1^2\alpha_2$ from (7.10) and noting (7.5) gives

* See the footnote in the beginning of §5.

$$\nabla_1^2\alpha_2 = \frac{h^2}{3!}(\nabla_1^6\tau_0 + \nabla_1^4\tau_2),$$

which upon substitution in the right-hand members of (7.9) and (7.10) gives the desired result:

$$\alpha_0 = \frac{h^4}{4!}f_1, \quad \alpha_2 = -\frac{h^2}{3!}f_1, \quad \alpha_4 = f_1,$$

where

$$f_1 \equiv \left(1 + \frac{h^2}{3}\nabla_1^2\right)(\nabla_1^4\tau_0 + \nabla_1^2\tau_2).$$

The expression for α_4 just given is obtained from (7.1).

It will be observed that (7.7) and (7.8) are given by (5.15) with $N=2$. It is easily checked with the aid of (7.1) and (7.6) that (5.14) is valid even when $N=2$.

The formulas for α_1, α_3, and α_5 are found to be

$$\alpha_1 = \frac{h^4}{5!}f_2, \quad \alpha_3 = -\frac{h^2}{10}f_2, \quad \alpha_5 = f_2,$$

where

$$f_2 = \left(1 + \frac{h^2}{5}\nabla_1^2\right)(\nabla_1^4\tau_1 + \nabla_1^2\tau_3).$$

It is easily checked that the equations (5.19) are valid when $M=2$.

8. **Critique of method.** The determination of Z_z, outlined above, leaves nothing to be desired from the standpoint of rigor, provided that the assumption that Z_z is adequately represented by a polynomial in z, of sufficiently high degree, is satisfied. The assumption that Z_z is expressible as an infinite series in powers of z leads to an infinite system of partial differential equations on the $a_k(x, y)$ of the form (4.6). This in turn will lead to infinite series developments for the a_k. An analysis of the behavior of the resulting series will prove exceedingly difficult, and is not attempted here. On the other hand, it will be recalled that a considerable portion of the theory of elasticity is built on the assumption that one is dealing with a class of functions which can be approximated arbitrarily closely by polynomials of sufficiently high degree. If it further be noted that a successful theory of moderately thick plates* has been developed on the hypothesis that Z_z is a polynomial of degree 3 in z, the assumption involved in (4.6) should be regarded as exceedingly

* Cf. A. E. H. Love, loc. cit., pp. 465–487.

mild. However, it must be noted that the choice of N is connected with the prescribed temperature function T via equations (5.14), (5.15), and (5.19). The significance of this connection becomes clear upon reflecting that one cannot hope to satisfy the differential equation (4.1), for an entirely arbitrary T, if he selects the solution for Z_z in the form (4.4) where n is small. For example, if an attempt is made to satisfy (4.1) by assuming a solution of the form (4.4) with $n=4$ or 5, then the polynomial solution will exist only for a class of temperature functions of the form (4.5) in which $T_k(x, y)=\tau_k/k!c$ satisfy the equations (5.14), (5.15), and (5.19). Obviously, the class of admissible temperature functions is greatly increased with the increase of the degree n in (4.4). In fact, if the $T_k(x, y)$ in (4.5) are polynomials in x and y, it is always possible to find a pair of numbers N and M so great that

$$\nabla_1^{2N+2}T_0 = 0, \qquad \nabla_1^{2N}T_2 = 0,$$
$$\nabla_1^{2M+2}T_1 = 0, \qquad \nabla_1^{2M}T_3 = 0.$$

Such a choice of N and M will, certainly, satisfy (5.14), (5.15), and (5.19) identically; hence the exact solution of the form (4.4) can be obtained. The case where the $T_k(x, y)$ are polynomials in x and y presents the most interesting distribution from the point of view of applications.

9. **Linear distribution of temperature.** It is of interest to investigate just how far the behavior of an elastic plate subjected to a nonuniform distribution of temperature departs from the state of generalized plane stress assumed by Nâdai and Marguerre. It will suffice to consider the case of linear distribution of temperature of the form

$$T = T_0(x, y) + zT_1(x, y).$$

The calculations will be carried out under the assumption that $\nabla_1^{10}T_0=0$ and $\nabla_1^8T_1=0$. Accordingly, the degree of the polynomial in z for Z_z in (4.4) is eight. This choice of T will certainly be satisfactory if T_0 and T_1 are polynomials in x and y of degrees 9 and 7, respectively.

Referring to (6.5) and (6.6) one finds easily that

$$A_4 = \frac{1}{4!}\nabla_1^4T_0, \qquad B_4 = -\frac{2}{4!}\nabla_1^4T_0, \qquad A_6 = \frac{1}{5!}\nabla_1^6T_0,$$

$$B_6 = -\frac{7}{3\cdot 5!}\nabla_1^6T_0, \qquad A_8 = \frac{41}{3\cdot 8!}\nabla_1^8T_0, \qquad B_8 = -\frac{37}{6\cdot 7!}\nabla_1^8T_0,$$

$$\overline{A}_4 = \frac{1}{5!}\nabla_1^4T_1, \quad \overline{B}_2 = -\frac{2}{5!}\nabla_1^4T_1, \quad \overline{A}_6 = \frac{22}{5\cdot 7!}\nabla_1^6T_1, \quad \overline{B}_4 = -\frac{54}{5\cdot 7!}\nabla_1^6T_1.$$

Hence from (5.16) and (5.18) it follows that

$$a_0 = \frac{\alpha E}{4!(1-\sigma)}\left(-h^4\nabla_1^4 T_0 - \frac{h^6}{5}\nabla_1^6 T_0 - \frac{41h^8}{5040}\nabla_1^8 T_0\right),$$

$$a_2 = \frac{\alpha E}{4!(1-\sigma)}\left(2h^2\nabla_1^4 T_0 + \frac{7h^4}{15}\nabla_1^6 T_0 + \frac{37h^6}{1260}\nabla_1^8 T_0\right),$$

$$a_1 = -\frac{\alpha E}{5!(1-\sigma)}\left(h^4\nabla_1^4 T_1 + \frac{11h^6}{105}\nabla_1^6 T_1\right),$$

$$a_3 = \frac{\alpha E}{5!(1-\sigma)}\left(2h^2\nabla_1^4 T_1 + \frac{9h^4}{35}\nabla_1^6 T_1\right).$$

The remaining functions a_j are readily calculated from (4.7); the results are

$$a_4 = -\frac{\alpha E}{4!(1-\sigma)}\left(\nabla_1^4 T_0 + \frac{h^2}{3}\nabla_1^6 T_0 + \frac{13h^4}{360}\nabla_1^8 T_0\right),$$

$$a_6 = \frac{\alpha E}{4!(1-\sigma)}\left(\frac{1}{15}\nabla_1^6 T_0 + \frac{h^2}{60}\nabla_1^8 T_0\right),$$

$$a_8 = -\frac{\alpha E}{4!(1-\sigma)}\frac{1}{560}\nabla_1^8 T_0,$$

$$a_5 = -\frac{\alpha E}{5!(1-\sigma)}\left(\nabla_1^4 T_1 + \frac{h^2}{5}\nabla_1^6 T_1\right),$$

$$a_7 = \frac{\alpha E}{5!(1-\sigma)}\frac{1}{21}\nabla_1^6 T_1.$$

Substituting these coefficients in (4.4) gives

$$(9.1)\quad \begin{aligned} Z_z = \frac{\alpha E}{4!(1-\sigma)}(z^2-h^2)^2\Bigg(&\nabla_1^4 T_0 - \frac{z}{5}\nabla_1^4 T_1 + \frac{z^2-3h^2}{15}\nabla_1^6 T_0 \\ &+ \frac{5z^3-11h^2z}{525}\nabla_1^6 T_1 - \frac{9z^4-66z^2h^2+41h^4}{7!}\nabla_1^8 T_0\Bigg). \end{aligned}$$

If n is assumed to be greater than 8, the additional terms appearing in the expression for Z_z do not affect the coefficients of the powers of ∇_1 just found.

It is clear that a similar analysis, pertaining to unheated plates, subjected to an arbitrary load, can be carried out. In this way one will arrive, from an altogether different point of view, at the solution of the problem discussed by Southwell.*

10. **Shearing stresses Z_x and Z_y.** The determination of the remaining stress components acting in the direction of the thickness of the plate will be

* R. V. Southwell, loc. cit.

given in this section, the point of interest being that it is possible to obtain these stresses directly from the fundamental equations and without imposing restrictions as severe as those implied by (2.8) and (2.9).

The equation (2.4) with $i=j=3$ gives

$$-\frac{\partial^2\Theta}{\partial z^2} = (1+\sigma)\nabla^2 Z_z + \frac{1+\sigma}{1-\sigma}\alpha E\nabla^2 T + \alpha E\frac{\partial^2 T}{\partial z^2}, \tag{10.1}$$

and from (2.5) one obtains

$$-\frac{\partial^2\Theta}{\partial z^2} = \nabla_1^2\Theta + \frac{2\alpha E}{1-\sigma}\nabla^2 T. \tag{10.2}$$

Equating (10.1) and (10.2) and simplifying gives

$$\nabla_1^2\Theta = (1+\sigma)\nabla^2 Z_z - \alpha E\nabla_1^2 T. \tag{10.3}$$

Setting $i=2$ and $j=3$ in equations (2.4) yields

$$\nabla^2 Z_y = -\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\partial z} - \frac{\alpha E}{1+\sigma}\frac{\partial^2 T}{\partial y\partial z}. \tag{10.4}$$

Taking ∇_1^2 of both sides of (10.4) and substituting from (10.3) furnishes the relation

$$\nabla^2\nabla_1^2 Z_y = -\nabla^2\frac{\partial^2 Z_z}{\partial y\partial z}. \tag{10.5}$$

Hence

$$\nabla_1^2 Z_y = -\frac{\partial^2 Z_z}{\partial y\partial z} + \phi(x, y, z), \tag{10.6}$$

where ϕ is a harmonic function.

But the last one of the conditions (4.2) demands

$$Z_y(x, y, \pm h) = 0,$$

so that

$$\nabla_1^2 Z_y = 0 \quad \text{on} \quad z = \pm h.$$

Thus, it follows from (10.6) that $\phi(x, y, z)$ satisfies the conditions

$$\phi(x, y, \pm h) = \left(\frac{\partial^2 Z_z}{\partial y\partial z}\right)_{z=\pm h}.$$

However, the function Z_z has been determined, and it follows from the foregoing that its form is

$$Z_z = (z^2 - h^2)^2 \sum_{j=0}^{k} a_j(x, y) z^j,$$

so that

$$\left(\frac{\partial^2 Z_z}{\partial y \partial z}\right)_{z=\pm h} = 0.$$

Thus the harmonic function $\phi(x, y, z)$ vanishes on the faces of the plate, and it follows that in a small region about the middle of the plate it cannot differ very much from zero. Hence, as a first approximation, one can choose $\phi = 0$. By assuming the solution for ϕ in the form

$$\phi = \sum_{j=0}^{n} \phi_j(x, y) z^j$$

and following the method of §4, one can improve upon the first approximation,* but it will be assumed, for the present, that $\phi = 0$ gives a satisfactory expression for $\nabla_1^2 Z_y$ in (10.6).

Therefore the conditions on Z_y become as follows:

$$\nabla_1^2 Z_y = -\frac{\partial^2 Z_z}{\partial y \partial z}, \tag{10.7}$$

where

$$Z_y = 0 \quad \text{on} \quad z = \pm h.$$

Observing that $\nabla^2 Z_y \equiv \nabla_1^2 Z_y + \partial^2 Z_y/\partial z^2$, and making use of (10.7), one is enabled to write (10.4) in the form

$$\frac{\partial^2 Z_y}{\partial z^2} = -\frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial y \partial z} - \frac{\alpha E}{1+\sigma}\frac{\partial^2 T}{\partial y \partial z} + \frac{\partial^2 Z_z}{\partial y \partial z}. \tag{10.8}$$

On the other hand, the differentiation of (3.2) with respect to y gives

$$\frac{\partial}{\partial y}\nabla_1^2 w = -2\alpha\frac{\partial^2 T}{\partial y \partial z} + \frac{\sigma - 1}{E}\frac{\partial^2 \Theta}{\partial y \partial z} - \frac{1+\sigma}{E}\frac{\partial^2 Z_z}{\partial y \partial z}. \tag{10.9}$$

Eliminating $\partial^2\Theta/\partial y \partial z$ between (10.8) and (10.9) produces

$$\frac{\partial^2 Z_y}{\partial z^2} = \frac{E}{1-\sigma^2}\frac{\partial}{\partial y}\nabla_1^2 w + \frac{\alpha E}{1-\sigma}\frac{\partial^2 T}{\partial y \partial z} + \frac{2-\sigma}{1-\sigma}\frac{\partial^2 Z_z}{\partial y \partial z}. \tag{10.10}$$

If it be assumed that $w(x, y, z)$ can be replaced by $w(x, y, 0)$, then the

* The process here will be much less involved since ϕ satisfies Laplace's equation.

right-hand member of (10.10) becomes a known function of x, y, and z, since the deflection of the middle plane, $z=0$, can be calculated from (3.7). It is to be noted that the assumption that the deflection of the middle surface be nearly the same as that of any plane parallel to the middle surface is not as severe as (2.8).

Thus, replacing w by w_0 in (10.10), and integrating twice with respect to z gives

$$Z_y = \frac{Ez^2}{2(1-\sigma^2)} \frac{\partial}{\partial y} \nabla_1^2 w_0 + \frac{\alpha E}{1-\sigma} \int_0^z \frac{\partial T}{\partial y} dz + \frac{2-\sigma}{1-\sigma} \int_0^z \frac{\partial Z_z}{\partial y} dz + z f_1 + f_2,$$

where f_1 and f_2 are arbitrary functions of x and y.

Since Z_y vanishes on $z = \pm h$, the functions f_1 and f_2 are uniquely determined. An elementary calculation shows that

$$\begin{aligned} Z_y = {} & \frac{E(z^2 - h^2)}{2(1-\sigma^2)} \frac{\partial}{\partial y} \nabla_1^2 w_0 + c_1 \int_0^z \frac{\partial T}{\partial y} dz + c_2 \int_0^z \frac{\partial Z_z}{\partial y} dz \\ & - \frac{z}{2h}\left(c_1 \int_{-h}^h \frac{\partial T}{\partial y} dz + c_2 \int_{-h}^h \frac{\partial Z_z}{\partial y} dz \right) \\ & - \frac{c_1}{2}\left(\int_0^h \frac{\partial T}{\partial y} dz + \int_0^{-h} \frac{\partial T}{\partial y} dz \right) \\ & - \frac{c_2}{2}\left(\int_0^h \frac{\partial Z_z}{\partial y} dz + \int_0^{-h} \frac{\partial Z_z}{\partial y} dz \right), \end{aligned} \tag{10.11}$$

where

$$c_1 = \frac{\alpha E}{1-\sigma}, \qquad c_2 = \frac{2-\sigma}{1-\sigma}.$$

The expression for Z_x can be obtained in precisely the same way. The resulting formula is (10.11) with y replaced by x.

The first term in the right-hand member of (10.11) is recognized as the expression for shear given by the ordinary thin plate theory under the hypotheses (2.8) and (2.9).

11. **Conclusion.** Some interesting deductions regarding the behavior of a heated elastic plate can be drawn immediately from (3.7). The right-hand member of (3.7) can be regarded as a fictitious normal load $p(x, y)$ which produces the same deflection in an unheated plate as that caused by the thermal stresses. Now, if the T_k are constants so that the temperature is a function of z alone, the right-hand member of (3.7) vanishes and w_0 satisfies

$$\nabla_1^4 w_0 = 0.$$

If the plate is clamped,

$$w_0 = \frac{\partial w_0}{\partial n} = 0$$

along the rigid support, and the only possible solution of the system is $w_0 = 0$. Hence a clamped plate under the action of thermal stresses alone will remain plane. The same conclusion is reached upon assuming a linear distribution of temperature of the form $T = T_0(x, y) + zT_1(x, y)$, where T is steady. However, a simply supported plate will buckle under the action of thermal stresses. An investigation pertaining to the action of such plates, based on the theory developed in this paper, will be published elsewhere.

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